A version of Hörmander's theorem in 2-smooth Banach spaces

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Abstract

We consider a stochastic evolution equation in a 2-smooth Banach space with a densely and continuously embedded Hilbert subspace. We prove that under Hörmander's bracket condition, the image measure of the solution law under any finite-rank bounded linear operator is absolutely continuous with respect to the Lebesgue measure. To obtain this result, we apply methods of the Malliavin calculus.

1 Introduction

Let E be a 2-smooth Banach space (below, we recall the definition), and $H \subset E$ be a Hilbert subspace. Further let H be dense in E, and the canonical embedding $H \hookrightarrow E$ be continuous. We consider the following stochastic evolution equation in E:

$$
\begin{aligned}
dX_t &= (AX_t + \alpha(X_t))dt + \sigma(X_t)dW_t, \\
X_0 &= x,
\end{aligned}
\tag{1}
$$

where W_t is an H-cylindrical Brownian motion, A is a generator of a strongly continuous semigroup on E, α is a function $E \to E$, and σ maps E to the space of γ-radonifying operators $H \to E$ (see [14]) denoted by $\gamma(H, E)$. Further let $\{e_i\}_{i=1}^{\infty}$ denote an orthonormal basis in H. We prove that if X_t is a solution to (1), $F : E \to \mathbb{R}^k$ is a bounded linear operator of rank k,

then, under Hörmander's bracket condition applied to the infinite system of vectors $\sigma_i(x) = \sigma(x)e_i$, $i = 1, 2, \ldots$, and $\sigma_0(x) = Ax + \alpha(x) + \sum_{i=1}^{\infty} \sigma_i'(x)\sigma_i(x)$, the law of FX_t for any fixed t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^k$. Since not every Banach space suits for consideration of equation (1), we work in 2-smooth Banach spaces, where we can employ the theory of stochastic integration and stochastic evolution equations [7, 6, 9, 10]. We mention that there exists a large class of UMD Banach spaces where the latter theory was developed as well (see [15, 16]). However, UMD Banach spaces do not seem suitable for proving our main result.

Regularity of transition probabilities for solutions to infinite-dimensional SDEs under Hörmander-type assumptions has been studied by many authors (see, for example, [2, 3, 11, 18]). Also, we would like to mention the work [4], where the authors prove the existence of the logarithmic derivative (see [1, 4]) for the transition probability of the solution to a Banach space valued SDE. However, all of the above articles, except [4], deal with Hilbert space-valued SDEs, and, to the author's knowledge, a Banach space version of Hörmander's theorem is obtained for the first time.

The paper is organized as follows: in Section 2 we sketch the proof of the existence and uniqueness of the mild solution to (1) by methods developed in [4]. In Section 3, we obtain an SDE for the Malliavin derivative of the solution to (1). The concept of the Malliavin derivative of a Banach space-valued random variable was introduced, for example, in [12]. In Section 4, we prove the Fréchet differentiability of the solution to (1) with respect to the initial data, and show that the Fréchet derivative is the unique solution to an SDE in $\gamma(H, E)$. The latter space is also 2-smooth which allows us to apply the results of Section 2 on the existence of solutions. In Section 5, under some additional assumptions, we prove the existence of the right inverse operator to the derivative from Section 4. Finally, in Section 6, we show the non-degeneracy of the Malliavin covariance matrix of FX_t, and, by this, the existence of a density of the law of FX_t with respect to the Lebesgue measure. In fact, we obtain an infinite-dimensional analog of Nualart's proof [13].

We remark that for an infinite dimensional SDE, the Fréchet derivative of the solution with respect to the initial data, in general, exists only in the mean-square sense, i.e. as a bounded linear operator $E \to L_2(\Omega, E)$, although it would be desirable for our construction to have it as bounded operator $E \to E$ a.s. In the finite dimensional case, the latter fact holds due to Kolmogorov's continuity theorem. Thus, one of the main difficulties of this

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work was to find assumptions under which the infinite-dimensional Fréchet derivative and its right inverse operator are a.s. bounded operators $E \to E$. To make our results valid for a larger class of operators A, such as Laplacian and other differential operators, we avoid the assumption on A to generate a group, as it was imposed in [3], although it would significantly simplify our arguments.

Finally, we remark, that the theory of differentiability of measures, developed in [1], offers an alternative, to the Malliavin calculus, approach to regularity of transition probabilities. This approach was undertaken in [4]. However, SDEs considered in [4] are not stochastic evolution equations, and therefore, the existence and smoothness of the density does not follow from [4] automatically. The present article considers the "traditional" Malliavin calculus approach to Hörmander's theorem.

2 Existence of the mild solution

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, W_t be an H-cylindrical Brownian motion, and $\mathcal{F}_t$ be the filtration generated by W_t. We consider stochastic evolution equation (1) in a 2-smooth Banach space E. We recall that a Banach space E is called 2-smooth if there exists a constant $C > 0$ so that for all x and y from E,

$$\|x + y\|^2 + \|x - y\|^2 \leqslant 2\|x\|^2 + C\|y\|^2.$$

We prove the existence of a mild solution to (1) on the interval $[0, T]$, $T > 0$, i.e. an $\mathcal{F}_t$-adapted stochastic process X_t satisfying

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}\alpha(X_s)\,ds + \int_0^t e^{(t-s)A}\sigma(X_s)\,dW_s, \qquad (2)$$

where $x \in E$, e^{tA} is the semigroup generated by A. We assume that $\alpha : E \to E$ and $\sigma : E \to \gamma(H, E)$ satisfy the following Lipschitz and linear growth conditions:

A1 $\|\alpha(x) - \alpha(y)\|_E + \|\sigma(x) - \sigma(y)\|_{\gamma(H,E)} \leqslant \gamma_1\|x - y\|_E$,
for all $x, y \in E$, and for some constant γ_1.

A2 $\|\alpha(x)\|_E + \|\sigma(x)\|_{\gamma(H,E)} \leqslant \gamma_2(1 + \|x\|_E)$,
for all $x, y \in E$, and for some constant γ_2.

For any Banach space G, in the space $\mathcal{F}_t$-adapted G-valued stochastic processes we introduce the norm:

$$\|\xi\|_{S_2(G)}^2 = \sup_{t \in [0,T]} \mathbb{E}\|\xi(t, \cdot)\|_G^2. \tag{3}$$

THEOREM 1. *Let A1 and A2 hold. Then equation* (2) *has a unique solution in the space $S_2(E)$. This solution has a continuous path modification.*

Proof. The scheme of the proof is similar to which was used in [4]. We will search for the solution in $S_2(E)$. Consider the map $\Gamma : S_2(E) \to S_2(E)$,

$$\Gamma(X_t) = e^{tA}x + \int_0^t e^{(t-s)A}\alpha(X_s)\,ds + \int_0^t e^{(t-s)A}\sigma(X_s)\,dW_s. \tag{4}$$

By the results of [7], $\int_0^t e^{(t-s)A}\sigma(X_s)\,dW_s$ is E-valued, and

$$\left\| \int_0^t e^{(t-s)A}\sigma(X_s)\,dW_s \right\|_E^2 \leqslant C \int_0^t \|e^{(t-s)A}\sigma(X_s)\|_{\gamma(H,E)}^2 ds.$$

By A2, the map Γ is well-defined. Then, A1 and usual stochastic integral estimates imply that there exists a constant $K > 0$ so that for each pair X and X' from $S_2(E)$

$$\sup_{t \in [0,T]} \mathbb{E}\|\Gamma^n(X_t) - \Gamma^n(X_t')\|_E^2 \leqslant \frac{K^n T^n}{n!} \sup_{t \in [0,T]} \mathbb{E}\|X_t - X_t'\|_E^2.$$

Pick up the integer n so that $\frac{K^n T^n}{n!} < 1$. Then $\Gamma^n : S_2 \to S_2$ is a contraction map. The unique fixed point of the map Γ^n is also the unique fixed point of Γ. By the results of [17], the stochastic convolution in (2) has a continuous version. This and Assumption A1 imply that the solution X_t also has a continuous version. $\square$

3   The Malliavin derivative of the solution

The Malliavin derivative of a Banach space-valued random variable was defined in [12], pp. 154-155. Let $\mathcal{H} = L_2([0,T], H)$ be the Hilbert space where we consider the isonormal Gaussian process $\mathbb{W}(h) = \int_0^t h(s)dW_s$, $h \in \mathcal{H}$.

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According to [12], the domain $\mathbb{D}^{1,2}$ of the Malliavin derivative operator D is defined by the squared norm

$$\|\xi\|_{\mathbb{D}^{1,2}}^2 = \|\xi\|_{L_2(\Omega,E)}^2 + \|D\xi\|_{L_2(\Omega,\gamma(\mathcal{H},E))}^2.$$

In the following, we will need the two lemmas below. Lemma 1 is proved in [19] (Lemma 3.7).

LEMMA 1. *Let G be a reflexive Banach space. Suppose $\xi_n \to \xi$ in $L_2(\Omega, G)$ and there is a constant $C > 0$ such that*

$$\sup_n \mathbb{E}\|D\xi_n\|_{\gamma(\mathcal{H},G)}^2 < C. \tag{5}$$

Then, $\xi \in \mathbb{D}^{1,2}$, $\mathbb{E}\|D\xi\|_{\gamma(\mathcal{H},G)}^2 < C$, and, moreover, there exists a weakly convergent subsequence $D\xi_{n_k} \to D\xi$.

Notice that a 2-smooth Banach space is uniformly smooth, and, therefore, reflexive. Hence, Lemma 1 holds with $G = E$. Lemma 2 below is a simple version of the chain rule.

LEMMA 2. *Let $\xi \in \mathbb{D}^{1,2}$ and let $F : E \to E$ have a bounded continuous Fréchet derivative. Then, $F(\xi) \in \mathbb{D}^{1,2}$, and*

$$DF(\xi) = F'(\xi)D\xi. \tag{6}$$

Proof. Take a sequence of smooth random variables ξ_n that converges to ξ in $\mathbb{D}^{1,2}$. Clearly, $DF(\xi_n) = F'(\xi_n)D\xi_n$. By boundedness of the derivative F', $F(\xi_n) \to F(\xi)$ in $L_2(\Omega, E)$. Moreover, $DF(\xi_n)$ satisfy Assumption (5) by the fact of convergence $D\xi_n \to D\xi$ in $L_2(\Omega, \gamma(\mathcal{H}, E))$, by the boundedness of $\|F'(\xi_n)\|_{\mathcal{L}(E)}$ uniformly in n, and by the ideal property of γ-radonifying operators. By Lemma 1, $F(\xi) \in \mathbb{D}^{1,2}$. Moreover, there is a subsequence ξ_{n_k} so that $DF(\xi_{n_k}) \to DF(\xi)$ weakly in $L_2(\Omega, \gamma(\mathcal{H}, E))$. On the other hand,

$$\|F'(\xi_{n_k})D\xi_{n_k} - F'(\xi)D\xi\|_{L_2(\Omega,\gamma(\mathcal{H},E))} \leqslant \|F'(\xi_{n_k})(D\xi_{n_k} - D\xi)\|_{L_2(\Omega,\gamma(\mathcal{H},E))}$$
$$+ \|(F'(\xi_{n_k}) - F'(\xi))D\xi\|_{L_2(\Omega,\gamma(\mathcal{H},E))}. \tag{7}$$

The first term on the right-hand side converges to zero by the boundedness of F' and by the ideal property of $\gamma(\mathcal{H}, E)$. As for the second term, we can find a further subsequence of ξ_{n_k}, we denote it again by ξ_{n_k}, that converges to ξ a.s. Then, the second term in (7) converges to zero by the boundedness of F', by Lebesgue's theorem, and, again, by the ideal property of $\gamma(\mathcal{H}, E)$. This proves (6). $\qquad\square$

We will need the next assumption.

A3 The functions $\alpha : E \to E$, $\sigma_i : E \to E$, $i = 1, \ldots, n$, have bounded Fréchet derivatives. Moreover, $\sigma' : E \to \gamma(H, \mathcal{L}(E))$ is continuous.

THEOREM 2. *Suppose A3 is fulfilled. Then, $X_t \in \mathbb{D}^{1,2}$ for all $t \in [0, T]$. Moreover, $DX_t \in L_2(\Omega \times [0, T], \gamma(H, E))$, and for $r \leqslant t$, $D_r X_t$ satisfies the following equation in $\gamma(H, E)$:*

$$D_r X_t = e^{(t-r)A} \sigma(X_r) + \int_r^t e^{(t-s)A} \alpha'(X_s) \, D_r X_s \, ds$$

$$+ \int_r^t e^{(t-s)A} \sigma'(X_s) \, D_r X_s \, dW_s. \quad (8)$$

For $r > t$, $D_r X_t = 0$.

Proof. First we note that $\gamma(H, E)$ is a 2-smooth Banach space, and, therefore, (8) is well-defined. We construct iterations by setting $X_t^{(0)} = x$, and $X_t^{(n+1)} = \Gamma(X_t^{(n)})$, where Γ is defined by (4). Notice that each successive iteration $X_t^{(n)}$ has a continuous version, since, by the results of [17], the stochastic convolution process has a continuous version. We are going to prove by induction on n that all successive iterations $X_t^{(n)}$ are in the domain $\mathbb{D}^{1,2}$. Clearly, $X_t^{(0)} \in \mathbb{D}^{1,2}$, and $DX_t^{(0)} = 0$. As the induction hypothesis, we assume the following: 1) $X_t^{(n)} \in \mathbb{D}^{1,2}$, 2) $DX_t^{(n)} \in L_2(\Omega \times [0, T], \gamma(H, E))$, 3) for each fixed $r > 0$ the path of $D_r X_t^{(n)}$ is uniformly continuous on $[r, T]$ in the mean-square sense, 4) $D_r X_t^{(n)} = 0$ for $r > t$, 5) $\mathbb{E} \|D_r X_t^{(n)}\|_{\gamma(H,E)}^4$ is bounded. Note that, by the induction hypothesis, we can evaluate $DX_t^{(n)}$ at any point $r \in [0, T]$, and write $D_r X_t^{(n)}$ for this evaluation. Let us prove these statements for $n + 1$. We start by showing the relation:

$$D_r X_t^{(n+1)} = e^{(t-r)A} \sigma(X_r^{(n)}) + \int_r^t e^{(t-s)A} \alpha'(X_s^{(n)}) D_r X_s^{(n)} \, ds$$

$$+ \int_r^t e^{(t-s)A} \sigma'(X_s^{(n)}) D_r X_s^{(n)} \, dW_s. \quad (9)$$

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For this, we need to prove that

$$D_r \int_0^t e^{(t-s)A}\sigma(X_s^{(n)})\, dW_s = e^{(t-r)A}\sigma(X_r^{(n)}) + \int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\, dW_s \tag{10}$$

and

$$D_r \int_0^t e^{(t-s)A}\alpha(X_s^{(n)})\, ds = \int_r^t e^{(t-s)A}\alpha'(X_s^{(n)})\, D_rX_s^{(n)}\, ds. \tag{11}$$

Note that the stochastic integral on the right-hand side of (10) is well-defined. Indeed, since $D_rX_s^{(n)}$ takes values in $\gamma(H,E)$, then, by A3, the integrand of the stochastic integral takes values in $\gamma(H,\gamma(H,E))$. This implies (see [14],[7]) that the stochastic integral in (10) is in $L_2(\Omega,\gamma(H,E))$, and, moreover, that there exists a constant $C > 0$ so that

$$\left\| \int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\, dW_s \right\|_{\gamma(H,E)}^2$$
$$\leqslant C \int_r^t \|e^{(t-s)A}\sigma'(X_s^{(n)})D_rX_s^{(n)}\|_{\gamma(H,\gamma(H,E))}^2\, ds.$$

To prove (10) and (11), suppose first that $r > t$. Fix a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = t\}$ and consider a simple integrand of the form

$$\sigma_N(X^{(n)},s) = \sum_{i=1}^N e^{(t-t_i)A}\sigma(X_{t_i}^{(n)})\,\mathbb{I}_{(t_{i-1},t_i]}(s). \tag{12}$$

Note that $\sigma_N(X^{(n)},s)$ converges to $\sigma(X_s^{(n)})$ in the mean-square sense which is implied by the uniform continuity of paths of $X_s^{(n)}$ in the $L_2(\Omega,E)$-norm. The latter uniform continuity is implied by the relation $X^{(n)} = \Gamma(X^{(n-1)})$, where Γ is defined by (4), and by the fact that $\mathbb{E}\|X_t^{(n)}\|_E^2$ is bounded uniformly in n and $t \in [0,T]$ which follows from the same relation and the usual stochastic integral estimates. Then, from Lemma 2 and from the equality $D_r(W_t e_i) = e_i\,\mathbb{I}_{[0,t]}(r)$, it follows that $D_r \int_0^t \sigma_N(X^{(n)},s)dW_s = 0$ if $D_rX_t^{(n)} = 0$. By taking the limit as the mesh of $\mathcal{P}$ goes to 0, we obtain that $D_r \int_0^t e^{(t-s)A}\sigma(X_s^{(n)})dW_s = 0$. Analogously, $D_r \int_0^t e^{(t-s)A}\alpha(X_s^{(n)})ds = 0$ if $D_rX_t^{(n)} = 0$. This proves that for $r > t$, $D_rX_t^{(n+1)} = 0$. Now take an $r \leqslant t$

and fix a partition $\mathcal{P} = \{0 = t_0 < t_1 < \cdots < t_N = t\}$ containing r. We have:

$$D_r \int_0^t \sigma_N(X^{(n)}, s)\, dW_s = e^{(t-r)A}\sigma(X_r^{(n)}) + \int_r^t D_r \sigma_N(X^{(n)}, s)\, dW_s, \quad (13)$$

where $D_r \sigma_N(X^{(n)}, s)$ is computed using (12). The right-hand side of the above relation, considered as a function of ω and r, converges to the right-hand side of (10) in $L_2(\Omega, \gamma(H, E))$ pointwise in $r \in [0, t]$. Indeed, there exists a constant $\gamma > 0$ so that

$$\mathbb{E}\Big\| \int_r^t e^{(t-s)A}\sigma'(X_s^{(n)})D_r X_s^{(n)}\, dW_s - \int_r^t D_r \sigma_N(X^{(n)}, s)\, dW_s \Big\|^2$$

$$\leqslant \gamma \Big[\Big(\sum_{i=1}^N \int_{t_{i-1}}^{t_i} \mathbb{E}\|e^{(t-s)A}\sigma'(X_s^{(n)}) - e^{(t-t_i)A}\sigma'(X_{t_i}^{(n)})\|^4 ds \Big)^{\frac{1}{2}} \Big(\int_r^t \mathbb{E}\|D_r X_s^{(n)}\|^4 ds \Big)^{\frac{1}{2}}$$

$$+ \sum_{i=1}^N \int_{t_{i-1}}^{t_i} \mathbb{E}\|D_r X_s^{(n)} - D_r X_{t_i}^{(n)}\|^2\, ds \Big].$$

The right-hand side of the above inequality converges to zero by the uniform continuity of paths of $X_s^{(n)}$, Lebesgue's theorem, and the induction hypothesis. The convergence of the right-hand side of (13) to the right-hand side of (10) holds also in $L_2(\Omega \times [0, T], \gamma(H, E))$, and, therefore, in $L_2(\Omega, \gamma(\mathcal{H}, E))$ by the canonical embedding of $L_2([0, T], \gamma(H, E))$ into $\gamma(\mathcal{H}, E)$ for type 2 Banach spaces (see [14]). Thus, equality (10) will be implied by Itô's isometry, by the continuity of paths, and by the closedness of the Malliavin derivative operator. Equality (11) follows from similar arguments. Therefore, $X_t^{(n+1)} \in \mathbb{D}^{1,2}$, $DX_t^{(n+1)} \in L_2(\Omega \times [0, T], \gamma(H, E))$, and relation (9) holds. This relation implies that the paths of $D_r X_t^{(n+1)}$ are continuous in the mean-square sense on $[r, T]$. The same relation and the maximal inequality for stochastic convolutions, proved in [17], imply that $\mathbb{E}\|D_r X_t^{(n+1)}\|^4$ is bounded. This completes the induction argument.

Now we would like to prove (5) for $\xi_n = X_t^{(n)}$. Relation (9) implies the estimate:

$$\mathbb{E}\|D_r X_t^{(n+1)}\|_{\gamma(H,E)}^2 \leqslant K\Big(1 + \int_r^t \mathbb{E}\|D_r X_s^{(n)}\|_{\gamma(H,E)}^2\Big),$$

where $K > 0$ is a constant which does not depend on r. This implies that for all n

$$\mathbb{E}\|D_r X_t^{(n)}\|_{\gamma(H,E)}^2 \leqslant K e^{KT}. \quad (14)$$

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Integrating (14) from 0 to T and using the fact of the canonical embedding of $L_2([0,T], \gamma(H,E))$ into $\gamma(\mathcal{H}, E)$ we obtain that $D_r X_t^n$ takes values in $\gamma(\mathcal{H}, E)$, and

$$\mathbb{E}\|DX_t^{(n)}\|_{\gamma(\mathcal{H},E)}^2 \leqslant J\,K\,\mathbb{E}\int_0^T \mathbb{E}\|D_r X_t^{(n)}\|_{\gamma(H,E)}^2\,dt \leqslant J\,K\,T\,e^{KT} \qquad (15)$$

where $J > 0$ is the embedding constant. By the results of Section 2, $X_t^{(n)} \to X_t$ in $L_2(\Omega, E)$. Hence, by Lemma 1, $X_t \in \mathbb{D}^{1,2}$, and, moreover, there is a weakly convergent subsequence $DX_t^{(n_k)} \to DX_t$. By (15), this subsequence contains a further subsequence which converges in $L_2(\Omega \times [0,T], \gamma(H,E)))$, say, to an element ζ. Then, again by the canonical embedding of $L_2([0,T], \gamma(H,E))$ into $\gamma(\mathcal{H}, E)$, $\zeta = DX_t$. The latter implies that that we can evaluate DX_t at $r \in [0,T]$, and, moreover, $D_r X_t$ takes values in $\gamma(H,E)$.

It remains to show (8). Take a $y' \in E^*$, and apply the functional y', and then the operator D_r, to the both parts of (2). Using the result from [8] on the Malliavin derivative of the stochastic integral (Proposition 5.4), for $r \leqslant t$ we obtain:

$$_E\langle D_r X_t, y'\rangle_{E^*} = {}_E\langle e^{(t-r)A}\sigma(X_r), y'\rangle_{E^*} + \int_r^t {}_E\langle e^{(t-s)A}\alpha'(X_s)\,D_r X_s, y'\rangle_{E^*}\,ds$$
$$+ \int_r^t {}_E\langle e^{(t-s)A}\sigma'(X_s)\,D_r X_s\,dW_s, y'\rangle_{E^*}.$$

This implies (8) since the above equation holds for all $y' \in E^*$, and (8) is a well-defined $\gamma(H,E)$-valued stochastic evolution equation. If $r > t$, then $D_r X_t = 0$ since it is the limit of $DX_t^{(n_k)}$ which is zero for $r > t$. $\qquad \square$

4 Differentiability with respect to the initial data

Consider the equation

$$Y_t = e^{tA} + \int_0^t e^{(t-s)A}\alpha'(X_s))Y_s\,ds + \int_0^t e^{(t-s)A}\sigma'(X_s)\,Y_s\,dW_s. \qquad (16)$$

which is obtained by formal differentiation of (2) with respect to the initial data, and is written with respect to the derivative operator Y_s. We would like

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to prove the existence of a solution to (16) in the space of bounded operators. We need the assumptions below.

A4 $\alpha'(x)$ is bounded in $\mathcal{L}(E)$ and $\gamma(H, E)$, and $\sigma'(x)$ is bounded in $\gamma(H, \gamma(H, E))$ and $\gamma(H, \mathcal{L}(E))$.

A5 The restriction of the semigroup e^{tA} to H is a semigroup on H.

For simplicity, we will use the same notations, i.e. $\alpha'(x)$, $\sigma'(x)$, e^{tA}, for the restrictions to H.

THEOREM 3. *Suppose A3, A4, and A5 are fulfilled. Then the solution $X_t(x)$ to* (2) *is Fréchet differentiable along H with respect to the initial data x. The derivative operator Y_t takes the form $Y_t = e^{tA} + V_t$ where V_t takes values in $\gamma(H, E)$. Moreover, Y_t is a solution to* (16), *and possesses a continuous path modification.*

Proof. First, consider the equation in E:

$$\xi_t = e^{tA}y + \int_0^t e^{(t-s)A}\alpha'(X_s)\xi_s\,ds + \int_0^t e^{(t-s)A}\sigma'(X_s)\,\xi_s\,dW_s. \qquad (17)$$

Since the derivatives $\alpha'(x)$ and $\sigma'(x)$ are bounded uniformly in $x \in E$, the proof of the existence of the solution and its continuous path modification is the same as in Theorem 1.

Consider the operator $V_t = Y_t - e^{tA}$, and rewrite (16) with respect to V_t:

$$V_t = \int_0^t e^{(t-s)A}\alpha'(X_s))V_s\,ds + \int_0^t e^{(t-s)A}\sigma'(X_s)\,V_s\,dW_s$$
$$+ \int_0^t e^{(t-s)A}\alpha'(X_s)\,e^{sA}\,ds + \int_0^t e^{(t-s)A}\sigma'(X_s)\,e^{sA}\,dW_s. \quad (18)$$

Note that by Assumptions 4 and 5 and by the ideal property of $\gamma(H, E)$, $e^{(t-s)A}\alpha'(X_s)\,e^{sA}$ takes values in $\gamma(H, E)$, and $e^{(t-s)A}\sigma'(X_s)\,e^{sA}$ takes values in $\gamma(H, \gamma(H, E))$. This, in particular, follows from the fact that $e^{(t-s)A}$, as a bounded operator $E \to E$, can be also regarded as a bounded operator $\gamma(H, E) \to \gamma(H, E)$ whose norm is not bigger than $\|e^{(t-s)A}\|_{\mathcal{L}(E)}$. Therefore, as in the proof of Theorem 2, the stochastic integral $\int_0^t e^{(t-s)A}\sigma'(X_s)\,e^{sA}\,dW_s$ takes values in $\gamma(H, E)$, and

$$\mathbb{E}\left\| \int_0^t e^{(t-s)A}\sigma'(X_s)\,e^{sA}\,dW_s \right\|_{\gamma(H,E)}^2 \leqslant C\,\mathbb{E}\int_0^t \|e^{(t-s)A}\sigma'(X_s)\,e^{sA}\|_{\gamma(H,\gamma(H,E))}^2\,ds,$$

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where $C > 0$ is a constant. Hence, the last two terms in (18) are bounded in $L_2(\Omega, \gamma(H, E))$. The existence of a solution to (18) can be proved in the space $S_2(\gamma(H, E))$ defined in Section 2, in exactly the same way as we proved the existence of the solution to (2). Moreover, the solution V_t to (18) is unique and possesses a continuous path modification. Equation (18) also implies that the process $Y_t = e^{tA} + V_t$ solves (16). Indeed, both terms $e^{(t-s)A}\sigma'(X_s) V_s$ and $e^{(t-s)A}\sigma'(X_s) e^{sA}$ take values in $\gamma(H, \gamma(H, E))$, and, therefore, the stochastic integral $\int_0^t e^{(t-s)A}\sigma'(X_s) Y_s \, dW_s$ is well-defined. Hence, Y_t verifies (16).

Now take a $y \in H$, and apply the both parts of (16) to y. We obtain that $Y_t y$ verifies (17). But the solution to (17) is unique in $S_2(E)$. From this and from the continuity of paths it follows that for all $t \in [0, T]$ $Y_t y = \xi_t$ a.s. Therefore, Y_t is the Fréchet derivative of $X_t(x)$ with respect to x along the space H. $\qquad\square$

5 The right inverse operator

In this section, under some additional assumptions, we prove the existence of the right inverse operator to Y_t. We will need Lemma 3 below, proved in [5].

LEMMA 3. *Let E, F, G be Banach spaces, and let $\{e_n\}$ be an orthonormal basis of H. Let $R \in \gamma(H, E)$, $S \in \gamma(H, F)$, and $T \in \mathcal{L}(E, \mathcal{L}(F, G))$. Then the sum*

$$\mathrm{Tr}\,_{R,S}T = \sum_{n=1}^{\infty}(TRe_n)(Se_n)$$

converges in G, does not depend on the choice of the orthonormal basis, and

$$\|\mathrm{Tr}\,_{R,S}T\|_G \leqslant \|T\|_{\mathcal{L}(E,\mathcal{L}(F,G))}\|R\|_{\gamma(H,E)}\|S\|_{\gamma(H,F)}.$$

Note that by this lemma, for each $x \in E$, the sum $\Sigma(x) = \sum_{i=1}^{\infty}\sigma_i'(x)\sigma_i'(x)$ converges in $\mathcal{L}(E, H)$ provided that $\sigma'(x) \in \gamma(H, \mathcal{L}(E, H))$. Indeed,

$$\|\Sigma(x)\|_{\mathcal{L}(E,H)} = \big\|\sum_{i=1}^{\infty}\sigma'(x)e_i\sigma'(x)e_i\big\|_{\mathcal{L}(E,H)} \leqslant \|\sigma'(x)\|_{\gamma(H,\mathcal{L}(E,H))}^2.$$

We will make additional assumptions:

A6 $e^{TA} : E \to E$ is an injectve map.

A7 There exists a Hilbert space $\tilde{H}$ containing E as a subspace, so that the canonical embedding of E into $\tilde{H}$ is continuous, and for all $x \in E$, $\sigma_i'(x)$, $i = 1, 2, \ldots$, can be extended to $\tilde{H}$. Moreover, each $\sigma_i'(x)$ maps $e^{tA}\tilde{H}$ to $e^{tA}H$ for all $t \in [0, T]$, and for some constant $C > 0$,

$$\|e^{-TA}\sigma'(x)e^{TA}\|_{\mathcal{L}_2(H, \mathcal{L}_2(\tilde{H}, H))} < C,$$

where $\mathcal{L}_2(H_1, H_2)$ denotes the space of the Hilbert-Schmidt operators from a Hilbert space H_1 to another Hilbert space H_2.

A8 For all $x \in E$, $\alpha'(x)$ maps $e^{tA}E$ to $e^{tA}H$ for all $t \in [0, T]$, so that

$$\|e^{-TA}\big(\Sigma(x) - \alpha'(x)\big)e^{TA}\|_{\mathcal{L}(E, H)} < C.$$

Since, by A6, all the operators e^{tA} are injective maps $E \to E$, as well as $H \to H$ due to A5, one can speak about the inverse operator e^{-tA}, in general unbounded, on $e^{tA}E$. Consider the equation

$$Z_t e^{tA} = I + \int_0^t Z_s\big(\Sigma(X_s) - \alpha'(X_s)\big)e^{sA}\,ds - \int_0^t Z_s\sigma'(X_s)e^{sA}\,dW_s \quad (19)$$

which is obtained by a formal derivation of an SDE for Y_t^{-1} and multiplying the both parts by e^{tA} from the right. Introducing the operator $R_t = Z_t e^{tA}$, we obtain the SDE for R_t:

$$R_t = I + \int_0^t R_s e^{-sA}\big(\Sigma(X_s) - \alpha'(X_s)\big)e^{sA}\,ds - \int_0^t R_s e^{-sA}\sigma'(X_s)e^{sA}\,dW_s. \tag{20}$$

THEOREM 4. *Let Assumptions A5–A8 be fulfilled. Then, equation* (20) *has a unique solution of the form* $R_t = I + U_t$ *where* U_t *is* $\mathcal{L}(E, H)$-*valued. Moreover, the operator* $Z_t = R_t e^{-tA}$, *defined on* $e^{tA}E$, *is the right inverse to* Y_t.

Proof. Written with respect to $U_t = R_t - I$, (20) takes the form:

$$U_t = \int_0^t U_s e^{-sA}\big(\Sigma(X_s) - \alpha'(X_s)\big)e^{sA}\,ds - \int_0^t U_s e^{-sA}\sigma'(X_s)e^{sA}\,dW_s$$

$$+ \int_0^t e^{-sA}\big(\Sigma(X_s) - \alpha'(X_s)\big)e^{sA}\,ds - \int_0^t e^{-sA}\sigma'(X_s)e^{sA}\,dW_s. \quad (21)$$

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Note that, for a Hilbert-Schmidt operator $B : \tilde{H} \to H$ we have the following relation between its different norms:

$$\|A\|_{\mathcal{L}(H)} \leqslant \|A\|_{\mathcal{L}(E,H)} \leqslant \|A\|_{\mathcal{L}_2(\tilde{H},H)}. \tag{22}$$

This allows us to solve (21) in $\mathcal{L}(E, H)$. Indeed, due to (22), the stochastic integral $\int_0^t e^{-sA}\sigma'(X_s)e^{sA}\, dW_s$ is well-defined in $\mathcal{L}_2(\tilde{H}, H)$, and, therefore, in $\mathcal{L}(E, H)$. Define the map $\Gamma : S_2(\mathcal{L}(E, H)) \to S_2(\mathcal{L}(E, H))$, $U \mapsto \Gamma(U)$, where $\Gamma(U)$ equals to the right-hand side of (21). For the stochastic integral in the first line of (21), we obtain:

$$\mathbb{E}\Big\|\int_0^t U_s e^{-sA}\sigma'(X_s)e^{sA}\, dW_s\Big\|^2_{\mathcal{L}(E,H)} \leqslant \mathbb{E}\Big\|\int_0^t U_s e^{-sA}\sigma'(X_s)e^{sA}\, dW_s\Big\|^2_{\mathcal{L}_2(\tilde{H},H)}$$
$$\leqslant \mathbb{E}\int_0^t \|U_s\|^2_{\mathcal{L}(E,H)}\|e^{-sA}\sigma'(X_s)e^{sA}\|^2_{\mathcal{L}_2(H,\mathcal{L}_2(\tilde{H},H))}\, ds.$$

Therefore, the stochastic integral $\int_0^t U_s e^{-sA}\sigma'(X_s)e^{sA}\, dW_s$ takes values in $\mathcal{L}(E, H)$. Moreover, the map Γ has a fixed point in $S_2(\mathcal{L}(E, H))$ which can be proved in exactly the same way as in Theorem 3, and hence, (21) has an $\mathcal{L}(E, H)$-valued solution. The solution U_t to (21) is also unique and possesses a continuous path modification. It is easy to verify that $R_t = I + U_t$ solves (20), and, moreover, it is a unique solution.

Let us consider the equation:

$$P_t = I + \int_0^t e^{-sA}\alpha'(X_s)e^{sA}P_s\, ds + \int_0^t e^{-sA}\sigma'(X_s)e^{sA}P_s\, dW_s. \tag{23}$$

Similar to (16), we can prove that (23) has a solution of the form $P_s = I + \tilde{V}_s$, where $\tilde{V}_s \in S_2(\gamma(H, E))$, and that $\tilde{V}_s$ is the unique solution to

$$\tilde{V}_t = \int_0^t e^{-sA}\alpha'(X_s)e^{sA}\tilde{V}_s\, ds + \int_0^t e^{-sA}\sigma'(X_s)e^{sA}\tilde{V}_s\, dW_s$$
$$+ \int_0^t e^{-sA}\alpha'(X_s)e^{sA}\, ds + \int_0^t e^{-sA}\sigma'(X_s)e^{sA}\, dW_s. \tag{24}$$

Multiplying (23) and (24) by e^{tA} from the left, we obtain (16) and (18), respectively. By the uniqueness of the solution to (18) in $S_2(\gamma(H, E))$, $V_t = e^{tA}\tilde{V}_t$ and, therefore, $Y_t = e^{tA}P_t$.

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Let us show that $P_t R_t = I$ on H. To compute $P_t R_t$, we apply Itô's formula to ${}_E\langle R_t y, P_t^* y'\rangle_{E^*}$, where $y \in H$, $y' \in E^*$:

$$
{}_E\langle P_t R_t y, y'\rangle_{E^*} = {}_E\langle y, y'\rangle_{E^*} + \int_0^t {}_E\langle e^{-sA}\alpha'(X_s)e^{sA}P_s R_s y, y'\rangle_{E^*} ds
$$

$$
+ \int_0^t {}_E\langle e^{-sA}\sigma'(X_s)e^{sA}P_s R_s y, y'\rangle_{E^*} dW_s - \int_0^t {}_E\langle P_s R_s e^{-sA}\sigma'(X_s)e^{sA}y, y'\rangle_{E^*} dW_s
$$

$$
+ \int_0^t {}_E\langle P_s R_s e^{-sA}\big(\Sigma(X_s) - \alpha'(X_s)\big)e^{sA}y, y'\rangle_{E^*} ds
$$

$$
- \int_0^t \sum_{k=1}^{\infty} {}_E\langle e^{-sA}\sigma_k'(X_s)e^{sA}P_s R_s e^{-sA}\sigma_k'(X_s)e^{sA}y, y'\rangle_{E^*} ds.
$$

Note that if we substitute $P_t R_t = I$, the above equation will be satisfied. Denoting $P_t R_t - I$ by Q_t we obtain the following SDE:

$$
Q_t = \int_0^t e^{-sA}\alpha'(X_s)e^{sA}Q_s ds + \int_0^t Q_s e^{-sA}(\Sigma(X_s) - \alpha'(X_s))e^{sA} ds
$$

$$
+ \int_0^t e^{-sA}\sigma'(X_s)e^{sA}Q_s \, dW_s - \int_0^t Q_s e^{-sA}\sigma'(X_s)e^{sA} \, dW_s
$$

$$
- \int_0^t \sum_{k=1}^{\infty} e^{-sA}\sigma_k'(X_s)e^{sA}Q_s e^{-sA}\sigma_k'(X_s)e^{sA} \, ds. \quad (25)
$$

By the assumptions imposed on α' and σ', the right-hand side takes values in $\gamma(H, E)$. Therefore, (25) is a well-defined SDE in $\gamma(H, E)$. The usual stochastic integral estimates and Gronwall's inequality imply that

$$
\mathbb{E}\|Q_t\|_{\gamma(H,E)}^2 = 0
$$

for all $t \in [0, T]$. Hence, $Q_t = 0$ a.s. This implies that $P_t R_t = I$ on H. By continuity of paths, the set $\tilde{\Omega} \subset \Omega$, $\mathbb{P}(\tilde{\Omega}) = 1$, where $P_t R_t = I$, can be choosen the same for all $t \in [0, T]$. Note that on H, $P_t = I - P_t U_t$. But $I - P_t U_t$ takes values in $\mathcal{L}(E)$ a.s. Therefore, P_t can be a.s. extended to a bounded operator $E \to E$. Thus, $P_t R_t = I$ everywhere on $\tilde{\Omega}$. This implies that $Y_t Z_t e^{tA}y = e^{tA}y$ a.s. for all $y \in E$. Hence, $Y_t Z_t = I$ a.s. on $e^{tA}E$. $\qquad\square$

6 A version of Hörmander's theorem

For every $x \in D(A)$, where $D(A)$ denotes the domain of A, we define $\sigma_0(x) = Ax + \alpha(x) - \frac{1}{2}\sum_{k=1}^{\infty}\sigma_k'(x)\sigma_k(x)$, and note that the third summand is well-defined. Indeed, application of Lemma 3 implies the convergence of the sum in E:

$$\Big\|\sum_{k=1}^{\infty}\sigma'(x)e_k\sigma(x)e_k\Big\|_E \leqslant \|\sigma'(x)\|_{\gamma(H,\mathcal{L}(E))}\|\sigma(x)\|_{\gamma(H,E)}.$$

SDE (1) takes the form

$$dX_t = \sigma_0(X_t)dt + \sum_{k=1}^{\infty}\sigma_k(X_t) \circ dW_t^k.$$

For two differentiable vector fields $V_1, V_2 : E \to E$ the Lie bracket $[V_1, V_2]$ is defined as in [13]. If a vector field of the form $A^k x$, $k = 1, 2, \ldots$, is involved in a Lie bracket, then A is formally treated as a bounded operator when we compute derivatives. For example, if $V : E \to D(A)$ is a vector field which is Fréchet differentiable $E \to E$, then, the Lie bracket $[Ax, V(x)] : D(A) \to E$ is computed by the formula

$$[Ax, V(x)] = AV(x) - V'(x)Ax.$$

For our version of Hörmander's theorem, we need Assumptions A9, A10, and H below:

A9 α, σ_i, $i = 1, 2, \ldots$, are infinitely differentiable functions $E \to D(A^{\infty})$, where $D(A^{\infty}) = \cap_{i=1}^{\infty}D(A^i)$; the function $\sigma' : E \to \gamma(H, \mathcal{L}(E))$ is differentiable.

A10 σ is a map $E \to \mathcal{L}(H, e^{TA}E)$, where the Banach space $e^{TA}E$ is equipped with the norm $\|x\|_{e^{TA}E} = \|e^{-TA}x\|_E$.

H (Hörmander's condition) The vector space spanned by the vector fields

$$\sigma_1, \sigma_2, \ldots, [\sigma_i, \sigma_j], [\sigma_i, [\sigma_j, \sigma_k]], i, j, k = 0, 1, \ldots$$

evaluated at point $x \in D(A^{\infty})$, is dense in E.

Note that under Assumption A9, all the Lie brackets in Assumption H are well-defined as vector fields $D(A^\infty) \to D(A^\infty)$. In the following, we will need Lemma 4 below.

LEMMA 4. *Let* $V : E \to E$ *be a* C^2*-vector field. Under Assumption A9, the term* $\left\{ [\sigma_0, V] + \frac{1}{2} \sum_{k=1}^\infty [\sigma_k, [\sigma_k, V]] \right\}(x)$, $x \in E$*, is well-defined. Moreover,*

$$-\frac{1}{2} \Big[\sum_{k=1}^\infty \sigma_k'(x)\sigma_k(x), V \Big](x) + \frac{1}{2} \sum_{k=1}^\infty [\sigma_k, [\sigma_k, V]](x)$$

$$= \sum_{k=1}^\infty \Big(-\sigma_k'(V'\sigma_k) + \frac{1}{2} V''\sigma_k\sigma_k + \sigma_k'(\sigma_k'V) \Big).$$

Proof. Let us compute the sum of two Lie brackets for a fixed k.

$$-\frac{1}{2} [\sigma_k'(x)\sigma_k(x), V] + \frac{1}{2} [\sigma_k, [\sigma_k, V]] = \frac{1}{2} \big(\sigma_k''\sigma_k V + \sigma_k'(\sigma_k'V) - V'(\sigma_k'\sigma_k)$$

$$+ \sigma_k'(\sigma_k'V) - \sigma_k'(V'\sigma_k) - \sigma_k''V\sigma_k - \sigma_k'(V'\sigma_k) + V''\sigma_k\sigma_k + V'(\sigma_k'\sigma_k) \big)$$

$$= -\sigma_k'(V'\sigma_k) + \frac{1}{2} V''\sigma_k\sigma_k + \sigma_k'(\sigma_k'V). \quad (26)$$

By Lemma 3, for the first term in the last line of (26) we have the estimate:

$$\Big\| \sum_{k=1}^\infty \sigma'(x)e_k(V'\sigma)(x)e_k) \Big\|_E \leqslant \|\sigma'(x)\|_{\gamma(H,\mathcal{L}(E))} \|V'(x)\sigma(x)\|_{\gamma(H,E)}.$$

For the third term in the last line of (26), we obtain:

$$\Big\| \sum_{k=1}^\infty \sigma'(x)e_k(\sigma'V)(x)e_k \Big\|_E \leqslant \|\sigma'(x)\|_{\gamma(H,\mathcal{L}(E))} \|(\sigma'V)(x)\|_{\gamma(H,E)}.$$

Finally, the estimate of the second term in the third line of (26) is:

$$\Big\| \sum_{k=1}^\infty [(V''\sigma)(x)e_k\sigma(x)e_k] \Big\|_E \leqslant \|V''(x)\|_{\mathcal{L}(E,\mathcal{L}(E))} \|\sigma(x)\|_{\gamma(H,E)}^2.$$

These estimates imply that we can take summations in k of the both parts in (26). The additional two estimates

$$\sum_{k=1}^\infty \Big\| \sigma''(x)V(x)e_k\sigma(x)e_k \Big\|_E \leqslant \|\sigma''(x)V(x)\|_{\gamma(H,\mathcal{L}(E))} \|\sigma\|_{\gamma(H,E)}$$

and

$$\Big\| \sum_{k=1}^{\infty} [(V'\sigma')(x)e_k \sigma(x)e_k] \Big\|_E \leqslant \|(V'\sigma')(x)\|_{\gamma(H,\mathcal{L}(E))} \|\sigma(x)\|_{\gamma(H,E)}$$

imply that the term $[\sigma_0, V]$ is well-defined. This, in turn, implies that the term $\frac{1}{2} \sum_{k=1}^{\infty} [\sigma_k, [\sigma_k, V]](x)$ is well-defined as well. $\qquad\square$

LEMMA 5. *Let X_t be a mild solution to SDE* (1) *with the initial condition $x \in D(A)$, and let A9 be fulfilled. Then X_t is also a strong solution to* (1).

Proof. The statement of the lemma can be verified by taking stochastic differentials of the both parts of (1). $\qquad\square$

LEMMA 6. *Let Assumptions A1, A2, and A10 be fulfilled. Further let $x \in D(A)$, and $V : E \to e^{TA}E$ be a vector field which is a C^2-smooth function $E \to E$. Then, if X_t is a strong solution to* (1) *and Z_t is the solution to* (19), *it holds that*

$$Z_t V(X_t) = V(x) + \sum_{k=1}^{\infty} \int_0^t Z_s[\sigma_k, V](X_s)\, dW_s^k$$
$$+ \int_0^t Z_s\big([\sigma_0, V] + \frac{1}{2} \sum_{k=1}^{\infty} [\sigma_k, [\sigma_k, V]]\big)(X_s)\, ds. \quad (27)$$

Proof. By Itô's formula (see [4], [7]),

$$V(X_t) = V(x) + \int_0^t V'(X_s)(AX_s + \alpha(X_s))ds + \int_0^t V'(X_s)\sigma(X_s)\, dW_s$$
$$+ \frac{1}{2} \int_0^t \sum_{k=1}^{\infty} V''(X_s)\sigma_k(X_s)\sigma_k(X_s)\, ds.$$

Equation (19) implies that for $x \in e^{TA}D(A)$,

$$Z_t x = x - \int_0^t Z_s A x\, ds + \int_0^t Z_s(\Sigma(X_s) - \alpha'(X_s))x\, ds - \int_0^t Z_s \sigma'(X_s)x\, dW_s.$$

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Applying Itô's formula to $Z_t V(X_t)$ we obtain:

$$Z_t V(X_t) = V(x) + \int_0^t Z_s \big(\Sigma(X_s) - A - \alpha'(X_s) \big) V(X_s) \, ds$$

$$+ \int_0^t Z_s \Big(V'(X_s)(AX_s + \alpha(X_s)) + \frac{1}{2} \sum_{k=1}^\infty V''(X_s)\sigma_k(X_s)\sigma_k(X_s) \Big) \, ds$$

$$- \int_0^t \sigma'(X_s) Z_s V(X_s) \, dW_s + \int_0^t Z_s V'(X_s)\sigma(X_s) \, dW_s$$

$$- \int_0^t \sum_{k=1}^\infty Z_s \sigma_k'(X_s) V'(X_s)\sigma_k(X_s) \, ds = V(x) +$$

$$\int_0^t Z_s [AX_s + \alpha(X_s), V(X_s)] \, ds + \sum_{k=1}^\infty \int_0^t Z_s[\sigma_k, V](X_s) \, dW_s$$

$$+ \int_0^t \sum_{k=1}^\infty Z_s[-\sigma_k'(V'\sigma_k) + \frac{1}{2} V''\sigma_k\sigma_k + \sigma_k'(\sigma_k'V)](X_s) \, ds.$$

By Lemma 4, the right-hand side of the above relation equals the right-hand side of (27). $\square$

The main result of this paper is the following version of Hörmander's theorem.

THEOREM 5. *Let Assumptions A1-A10 and Hörmander's condition H be satisfied. Further let $F : E \to \mathbb{R}^k$ be a bounded linear operator of rank k. Then, for any fixed t, the probability distribution of FX_t is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^k$.*

Proof. In Theorem 2 we proved that the Malliavin derivative $D_r X_t$, where $r \leqslant t$ are fixed, verifies equation (8). The uniqueness of the solution to (8) in $S_2(\gamma(H, E))$ follows from the results of Section 4. Let us note that the process $Y_t Z_r \sigma(X_r)$ also satisfies (8). Indeed, (16) implies:

$$Y_t = e^{(t-r)} Y_r + \int_r^t e^{(t-s)A} \alpha'(X_s) Y_s \, ds + \int_0^t e^{(t-r)A} \sigma'(X_s) Y_s \, dW_s.$$

Noticing that $Y_t Z_r \sigma(X_r)$ takes values in $\gamma(H, E)$, we multiply the both sides of the above equation by $Z_r \sigma(X_r)$. Taking into consideration that

$Y_r Z_r \sigma(X_r) = \sigma(X_r)$, we obtain that $Y_t Z_r \sigma(X_r)$ verifies (8). By uniqueness of the solution to (8) in $S_2(\gamma(H, E))$, $D_r X_t = Y_t Z_r \sigma(X_r)$.

Fix a $t > 0$. Let X_s be the solution of (1), and let $\xi_s = F X_s$. By Lemma 2, the Malliavin derivative $D_r \xi_t$, $r < t$, equals to

$$D_r \xi_t = F Y_t Z_r \sigma(X_r). \tag{28}$$

Further let γ_t denote the Malliavin covariance matrix of ξ_t. Using relation (28) we can write down γ_t in the form:

$$\gamma_t = (F \circ Y_t) C_t (F \circ Y_t)^*$$

where the operator $C_t : E^* \to E$ is defined as

$$C_t = \int_0^t Z_r \sigma(X_r) \sigma(X_r)^* Z_r^* \, dr.$$

By Theorem 2.1.2. of [13], the statement of the theorem will be implied by the invertibility of γ_t. In order to show that $(\gamma_t x, x)_{\mathbb{R}^k} > 0$ for all $x \in \mathbb{R}^k$, it suffices to prove that for all $\varphi \in E^*$, $\varphi \neq 0$,

$$_E \langle C_t \varphi, \varphi \rangle_{E^*} > 0 \tag{29}$$

with probability one. Indeed,, for every $x \in \mathbb{R}^k$,

$$(\gamma_t x, x)_{\mathbb{R}^k} = {}_E \langle C_t Y_t^* F^* x, Y_t^* F^* x \rangle_{E^*}.$$

Note that $\ker F^* = \{0\}$. Indeed, assume that there is a $z \in \mathbb{R}^k$, $z \neq 0$, such that $F^* z = 0$. Then, for all $y \in E$, $(Fy, z)_{\mathbb{R}^k} = {}_E \langle y, F^* z \rangle_{E^*} = 0$. Thus, z is orthogonal to $\operatorname{Im} F$ in $\mathbb{R}^k$ which contradicts to the assumption that F has rank k. Also, note that $\ker Y_t^* = \{0\}$. Indeed, $Y_t^* y' = 0$ implies $Z_t^* Y_t^* y' = y' = 0$. Hence, $Y_t^* F^* x \in E^*$ is non-zero if $x \in \mathbb{R}^k$ is non-zero.

Thus we have to prove (29). Let us assume that there exists a $\varphi_0 \neq 0$ such that

$$P\{_E \langle C_t \varphi_0, \varphi_0 \rangle_{E^*} = 0\} > 0. \tag{30}$$

Take a $\varphi \in E^*$. We have:

$$_E \langle C_t \varphi, \varphi \rangle_{E^*} = \int_0^t \|\sigma(X_r)^* Z_r^* \varphi\|_H^2 \, dr = \int_0^t \sum_{k=1}^\infty (e_k, \sigma(X_r)^* Z_r^* \varphi)_H^2 \, dr$$

$$= \int_0^t \sum_{k=1}^\infty {}_E \langle Z_r \sigma_k(X_r), \varphi \rangle_{E^*}^2 \, dr. \tag{31}$$

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Define random spaces $K_s \subset E$:

$$K_s = \overline{\text{span}\{Z_\zeta \sigma_k(X_\zeta); \ \zeta \in [0, s], k \in \mathbb{N}\}}.$$

The family of vector spaces $\{K_s, s \geqslant 0\}$ is increasing. Let $K_{0+} = \cap_{s>0} K_s$. By the Blumental zero-one law, K_{0+} is deterministic with probability one, since every random variable $b \in K_{0+}$ is constant with probability one. Let $N > 0$ be an integer, and let N_s be the codimension of K_{0+} in K_s, possibly infinite. Consider the non-decreasing adapted process $\{\min\{N, N_s\}, s > 0\}$, and the stopping time

$$S = \inf\{s > 0 : \ \min\{N, N_s\} > 0\}.$$

Note that $\mathbb{P}\{S > 0\} = 1$. Indeed, if we assume that $\mathbb{P}\{S = 0\} > 0$, it would imply that with a positive probability the codimension of K_{0+} in K_s is positive for any $s > 0$. The latter fact implies that with a positive probability the codimension of K_{0+} in $\cap_{s>0} K_s$ is positive as well, which is a contradiction.

Next, note that $K_{0+} \neq E$. Indeed, if $K_{0+} = E$, then $K_s = E$ for all $s > 0$. Therefore, if $\varphi \in E^*$ is such that $_E\langle C_t \varphi, \varphi \rangle_{E^*} = 0$, then $_E\langle Z_r \sigma_k(X_r), \varphi \rangle_{E^*} = 0$ for all $r \in [0, t]$ and for all $k \in \mathbb{N}$ by (31). This implies that φ is zero on K_s, and hence, $\varphi = 0$, which contradicts to hypothesis (30).

Take a non-zero functional $\varphi \in E^*$ containing K_{0+} in its kernel. Note that for all $s < S$, $\varphi(K_s) = 0$, and hence,

$$_E\langle Z_s \sigma_k(X_s), \varphi \rangle_{E^*} = 0 \quad \text{for all } k \text{ and } s < S. \tag{32}$$

Introduce the following sets of vector fields:

$$\begin{aligned} \Sigma_0 &= \{\sigma_1, \sigma_2, \dots, \sigma_k, \dots\} \\ \Sigma_n &= \{[\sigma_0, V], [\sigma_k, V], k \in \mathbb{N}, V \in \Sigma_{n-1}\} \\ \Sigma &= \cup_{n=1}^\infty \Sigma_n \end{aligned}$$

and

$$\begin{aligned} \Sigma'_0 &= \Sigma_0, \\ \Sigma'_n &= \{[\sigma_k, V], k = 1, 2, \dots, V \in \Sigma'_{n-1}; \\ &\qquad [\sigma_0, V] + \frac{1}{2} \sum_{k=1}^\infty [\sigma_k, [\sigma_k, V]], V \in \Sigma'_{n-1}\} \\ \Sigma' &= \cup_{i=1}^\infty \Sigma'_n. \end{aligned}$$

Let $\Sigma_n(x)$ and $\Sigma'_n(x)$ denote the subspaces of E obtained from Σ_n and Σ'_n, respectively, by evaluating the vector fields of the latters at point $x \in D(A^\infty)$. Note that the vector fields from Σ_n (resp. Σ'_n) are well-defined on $D(A^{n-1})$. Clearly, the spaces spanned on $\Sigma(x)$ and $\Sigma'(x)$ coincide with each other. They also coincide with the space E by Assumption H. Let us show that

$$\varphi(\Sigma'_n(x)) = 0 \quad \text{for all } n. \tag{33}$$

By Assumption H, this will imply that $\varphi = 0$, and hence, it will be a contradiction. Property (33) is implied by the following stronger property:

$$_E\langle Z_s V(X_s), \varphi \rangle_{E^*} = 0 \quad \text{for all } s < S,\ V \in \Sigma'_n,\ n \geqslant 0. \tag{34}$$

We show (34) by induction on n. For $n = 0$, (34) follows from (32). We assume that (34) holds for $n - 1$, and show that it holds for n. Let $V \in \Sigma'_{n-1}$. Note that if V takes values in $e^{tA}E$, then $[\sigma_k, V]$ and $[\sigma_0, V] + \frac{1}{2}\sum_{k=1}^\infty [\sigma_k, [\sigma_k, V]]$ also take values in $e^{tA}E$. By Lemma 6,

$$0 = {}_E\langle Z_s V(X_s), \varphi \rangle_{E^*} = {}_E\langle V(x), \varphi \rangle_{E^*} + \sum_{k=1}^\infty \int_0^s {}_E\langle Z_r[\sigma_k, V](X_r), \varphi \rangle_{E^*} dW_r^k$$

$$+ \int_0^s {}_E\Big\langle Z_r\Big\{[\sigma_0, V] + \frac{1}{2}\sum_{k=1}^\infty [\sigma_k, [\sigma_k, V]]\Big\}(X_r), \varphi \Big\rangle_{E^*} dr$$

which holds for all $s < S$. Since $_E\langle V(x), \varphi \rangle_{E^*} = 0$, it implies that for all $s < S$, the quadratic variation of the martingale part and the bounded variation part of this semimartingale must be zero. This proves (34). $\qquad\square$

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